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DEPARTMENT OF MATHEMATICS

STUDIES IN EIGENVALUE PROBLEMS

Technical Report 15

CHARACTERIZATION OF POSITIVE REPRODUCING KERNELS.  
APPLICATIONS TO GREEN'S FUNCTIONS.

by

N. Aronszajn and K. T. Smith

FC

Paper written under Contract Nonr 58304

Lawrence, Kansas

1956

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1. Introduction.

The object of this paper is to prove a theorem which characterizes the proper functional Hilbert spaces whose reproducing kernels are positive. Since certain Green's functions are among the most important examples of reproducing kernels, it is natural that the general theorem leads to results about the positiveness of Green's functions.

Not all Green's functions are reproducing kernels: those and only those are, which correspond to positive definite differential problems of sufficiently high order. This restriction on the order is avoided by use of the notion of pseudo-reproducing kernels for general functional Hilbert spaces. If the Green's function for a real elliptic self-adjoint linear differential problem exists, then it is a pseudo-reproducing kernel if and only if the problem is positive definite. Details of the underlying theory of pseudo-reproducing kernels are not given, except for those basic definitions and properties which are necessary to make clear the application of the abstract theorem about positiveness.

In the case of second order problems the question of which Green's functions are positive is answered completely by a proof of the equivalence of the following statements: (i) the problem is positive definite; (ii) the Green's function exists and is a pseudo-reproducing kernel; (iii) the Green's function exists and is positive. In

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1. Paper written under contract with Office of Naval Research N58304.

the case of higher order problems well-known examples show that these statements are not equivalent: (iii) does not follow from (i) and (ii) even for the square of the Laplacian and the boundary conditions  $u = \frac{\partial u}{\partial \nu} = 0$ .<sup>1</sup> However, we give examples of fourth order problems where it can be proved by the abstract theorem that the Green's function is positive, and where a simple direct proof does not seem to lie near at hand. The results are not restricted to problems about domains in Euclidean space. They are stated and proved for problems about relatively compact domains in oriented differentiable manifolds.

Our results on second order problems include as quite special cases some of the results of Bergman and Schiffer [6]<sup>2</sup>, and can be proved, in a heuristic manner at least, by following the lines set forth by Bergman and Schiffer in the special cases. The difficulty in this approach lies in the lack of general information about the sets of points on which solutions to the differential equation are zero. Our method does not require information of this kind.

## 2. Reproducing kernels.

The main theorem of the paper, Theorem 1 below, gives necessary and sufficient conditions in order that the reproducing

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1. R. J. Duffin [7], P. R. Garabedian [8].

2. These authors consider only the differential operator  $-\Delta + p$ ,  $p > 0$ , and consider boundary conditions slightly more special than ours; they consider explicitly only the case of 2 variables. However, they treat two questions related to the question treated here: when a Green's function is positive, and when one Green's function is larger than another. The second question is not discussed in this paper, but will be discussed from a similar general point of view in another paper.

kernel  $K(x,y)$  of a proper functional Hilbert space be non-negative.<sup>1.</sup> It is proved in [2] that condition (a) in the theorem below is necessary and sufficient in order that  $K(x,y)$  be real.

Theorem 1. In order that the reproducing kernel  $K(x,y)$  of the proper functional Hilbert space  $\mathcal{F}$  be non-negative it is necessary and sufficient that  $\mathcal{F}$  have the two properties

- (a) If  $u \in \mathcal{F}$ , then  $\bar{u} \in \mathcal{F}$  and  $\|\bar{u}\| = \|u\|$ .
- (b) For each real-valued  $u \in \mathcal{F}$  there exists  $\tilde{u} \in \mathcal{F}$  such that  $\tilde{u}(x) \geq |u(x)|$  for all  $x$  and  $\|\tilde{u}\| \leq \|u\|$ .

Proof. Assuming that  $\mathcal{F}$  has the properties (a) and (b), let  $u \in \mathcal{F}$ , let  $\tilde{u}$  be such that (b) holds, and set  $u^+ = \frac{1}{2}(\tilde{u} + u)$  and  $u^- = \frac{1}{2}(\tilde{u} - u)$ . Clearly  $u^+$  and  $u^-$  are both non-negative, and  $\tilde{u} = u^+ + u^-$  and  $u = u^+ - u^-$ . By using (a) it is proved easily that the scalar product of real-valued functions is real. Therefore, from  $\|\tilde{u}\| \leq \|u\|$  it follows that  $(u^-, u^+) \leq 0$ , and hence that  $(u^-, u) = (u^-, u^+) - (u^-, u^-) \leq -\|u^-\|^2$ . By virtue of the result from [2] quoted above,  $K_y(x) = K(x,y)$  is real-valued for every  $y$  so that this inequality can be applied to  $u = K_y$  to give  $0 \leq K_y^-(y) = (K_y^-, K_y) \leq -\|K_y^-\|^2$ . This is possible only if  $K_y^- = 0$  so that

1. A proper functional Hilbert space is a Hilbert space whose elements are functions on a basic set  $\mathcal{E}$ , such that the value of a function at any point in  $\mathcal{E}$  is a continuous linear functional. To each proper functional Hilbert space  $\mathcal{F}$  there corresponds a function  $K(x,y)$ , called the reproducing kernel of  $\mathcal{F}$ , defined on  $\mathcal{E} \times \mathcal{E}$  and having the properties:

- (a) for each  $y \in \mathcal{E}$ , the function  $K_y(x) = K(x,y)$  belongs to  $\mathcal{E}$ .
- (b) the reproducing property: for each  $u \in \mathcal{F}$ ,  $u(y) = (u, K_y)$ .

$$K_y = K_y^+ \geq 0.$$

Assuming that  $K(x,y) \geq 0$ , for each real-valued  $u \in \mathcal{F}$  let  $u'$  and  $u''$  be the projections of  $u$  and of  $-u$  on the closed convex cone with vertex 0 generated by the set  $\{K_y\}$ . Then for every  $\rho \geq 0$  and every  $v$  in the cone,  $\|u-u'\|^2 \leq \|u-u'-\rho v\|^2$ , from which it follows that for every  $v$  in the cone  $(u'-u, v) \geq 0$ . Applying this to  $v = K_y$ , we obtain  $u'(y) \geq u(y)$  for every  $y$ . Similarly,  $u''(y) \geq -u(y)$  for every  $y$ . Since  $K(x,y)$  is non-negative, both  $u'$  and  $u''$  are non-negative, so that if  $\tilde{u} = u' + u''$ , then  $\tilde{u}(x) \geq |u(x)|$  for all  $x$ . All that remains is to show that  $\|\tilde{u}\| \leq \|u\|$ . This was done in another paper [5], but since the proof is very short it bears repetition.

If  $\varphi'$ ,  $\alpha$ , and  $\varphi''$  are the angles between  $u$  and  $u'$ ,  $u'$  and  $u''$ , and  $u''$  and  $-u$ , respectively, then, as  $u - u'$  is orthogonal to  $u'$  and  $-u - u''$  is orthogonal to  $u''$ , the inequality to be proved takes the form  $\|u\|^2(\cos^2 \varphi' + \cos^2 \varphi'' + 2\cos \varphi' \cos \varphi'' \cos \alpha) \leq \|u\|^2$ , which is easy to establish with the aid of the inequalities  $0 \leq \varphi' \leq \frac{\pi}{2}$ ,  $0 \leq \varphi'' \leq \frac{\pi}{2}$ , and  $\pi \leq \varphi' + \alpha + \varphi''$ .

Remark 1. If condition (a) holds for a set of functions which is dense in  $\mathcal{F}$ , then condition (a) holds for all functions in  $\mathcal{F}$ . If condition (b) holds for a set of functions which is dense in the set of all real-valued functions in  $\mathcal{F}$ , then condition (b) holds for all real-valued functions in  $\mathcal{F}$ .

Remark 2. The following statement about matrices is a consequence of Theorem 1. Let  $\{a_{ij}\}$  be a real positive definite



matrix. In order that every element in  $\{a_{ij}\}^{-1}$  be non-negative it is necessary and sufficient that for each real vector  $u$  there exist  $\tilde{u}$  such that  $\tilde{u}_i \geq |u_i|$  for all  $i$  and  $\sum a_{ij} \tilde{u}_i \tilde{u}_j \leq \sum a_{ij} u_i u_j$ .

### 3. Pseudo-reproducing kernels.

The functional spaces which are of use in differential problems are proper functional Hilbert spaces in problems of order larger than the dimension of the underlying Euclidean space. In problems of smaller order the spaces are more general functional Hilbert spaces in which the functions are not defined everywhere.

A functional space is a normed linear class  $\mathcal{F}$  of functions on a basic set  $\mathcal{E}$ , each defined except on some exceptional set belonging to a hereditary  $\sigma$ -ring  $\mathcal{A}$  (the exceptional class of sets).

It is assumed that the norm in  $\mathcal{F}$  has the property that if

$\|u_n - u\| \rightarrow 0$ , then, for some subsequence  $\{u_{n_k}\}$ ,  $u_{n_k}(x) \rightarrow u(x)$  except on a set in  $\mathcal{A}$ . A complete discussion of functional spaces and functional completion can be found in [5].

Let  $\mu$  be a  $\sigma$ -finite complete measure on a set  $\mathcal{E}$ . The class of measurable subsets of  $\mathcal{E}$  of finite measure will be denoted by  $\dot{\mathcal{E}}$ , the sets in  $\dot{\mathcal{E}}$  by  $\dot{x}$ ,  $\dot{y}$ , etc. A  $\mu$ -measurable functional Hilbert space is a functional Hilbert space  $\mathcal{F}$  on the basic set  $\mathcal{E}$  such that: (a) all the exceptional sets are of measure 0, (b) each  $u \in \mathcal{F}$  is integrable on every  $\dot{x} \in \dot{\mathcal{E}}$ , (c) if  $\|u\| \neq 0$ , then  $\int_{\dot{x}} u d\mu \neq 0$  for some  $\dot{x} \in \dot{\mathcal{E}}$ , and (d) for every  $\dot{x} \in \dot{\mathcal{E}}$ ,  $\int_{\dot{x}} u d\mu$  is a continuous linear functional of  $u$ .<sup>1</sup>

1. The definition is valid for general incomplete spaces. In case

A  $\mu$ -measurable functional Hilbert space  $\mathcal{F}$  determines a proper functional Hilbert space  $\dot{\mathcal{F}}$  whose basic set is the class  $\dot{\mathcal{E}}$ .  $\dot{\mathcal{F}}$  consists of all functions  $\dot{u}$  to which there corresponds some  $u \in \mathcal{F}$  such that  $\dot{u}(\dot{x}) = \int_{\dot{x}} u d\mu$  for every  $\dot{x} \in \dot{\mathcal{E}}$ . The mapping  $u \rightarrow \dot{u}$  is one to one, and when  $(\dot{u}, \dot{v})$  is defined to be  $(u, v)$ , it becomes a Hilbert space isomorphism between  $\mathcal{F}$  and  $\dot{\mathcal{F}}$ . The reproducing kernel for  $\dot{\mathcal{F}}$  is the function  $\dot{K}(\dot{x}, \dot{y}) = (u_{\dot{y}}, u_{\dot{x}})$ , where  $u_{\dot{x}} \in \mathcal{F}$  is such that for every  $v \in \mathcal{F}$ ,  $\int_{\dot{x}} v d\mu = (u, u_{\dot{x}})$ .

A pseudo-reproducing kernel for a  $\mu$ -measurable functional Hilbert space  $\mathcal{F}$  is a function  $K(x, y)$  with the property that: for every  $\dot{x}$  and  $\dot{y}$  in  $\dot{\mathcal{E}}$ ,  $K(x, y)$  is integrable over  $\dot{x} \times \dot{y}$ , and  $\dot{K}(\dot{x}, \dot{y}) = \int_{\dot{x}} \int_{\dot{y}} K(x, y) d\mu(y) d\mu(x)$ . It is clear that this condition determines  $K(x, y)$  a.e. on  $\mathcal{E} \times \mathcal{E}$ .<sup>1</sup>

Since  $\dot{K}(\dot{x}, \dot{y})$  is an additive function of the rectangle  $\dot{x} \times \dot{y}$ , it can be extended to an additive set function on the ring generated by the rectangles, so that by the Radon-Nikodym theorem, a necessary and sufficient condition in order that the  $\mu$ -measurable functional Hilbert space  $\mathcal{F}$  have a pseudo-reproducing kernel is that the function  $\dot{K}(\dot{x}, \dot{y})$  be an absolutely continuous function of

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of a complete space, in particular in case of a Hilbert space, (d) follows from (a), (b), (c):

1. The beginnings of the theory of measurable spaces and pseudo-reproducing kernels are in the notes [1]. These notions can be defined in an absolute way, independent of a measure given beforehand. The relative notions defined here are sufficient for our needs in the present paper and avoid some of the difficulties inherent in the absolute notions.

the rectangle  $\dot{x} \times \dot{y}$ .

If a measure  $\mu$  and a function  $K(x,y)$  integrable over all  $\dot{x} \times \dot{y}$  are given, then a necessary and sufficient condition in order that  $K(x,y)$  be a pseudo-reproducing kernel for some  $\mu$ -measurable functional Hilbert space  $\mathcal{F}$  is that the function  $\dot{K}(\dot{x}, \dot{y}) = \int_{\dot{x}} \int_{\dot{y}} K(x,y) d\mu(y) d\mu(x)$  be a positive matrix on  $\dot{E}$ . If  $\dot{K}(\dot{x}, \dot{y})$  is a positive matrix,  $\mathcal{F}$  is constructed as follows. Let  $\mathcal{F}_0$  be the class of all linear combinations  $u = \sum a_i u_{\dot{y}_i}$ , where  $u_{\dot{y}_i}(x) = \int_{\dot{y}_i} K(x,y) d\mu(y)$ , with the norm

$$(3.1) \quad \|u\|^2 = \sum a_i \bar{a}_j K(\dot{y}_j, \dot{y}_i).$$

It can be proved that  $\mathcal{F}_0$  has a functional completion  $\mathcal{F}$ , which is a  $\mu$ -measurable functional Hilbert space.<sup>1</sup> Obviously,  $K(x,y)$  is a pseudo-reproducing kernel for  $\mathcal{F}$ .

The proper generalization of Theorem 1 to  $\mu$ -measurable functional Hilbert spaces and pseudo-reproducing kernels is clear.

If we put  $f = \sum a_i \chi_{\dot{y}_i}$ , where  $\chi_{\dot{y}_i}$  is the characteristic function of  $\dot{y}_i$ , then (3.1) reads

$$(3.2) \quad \text{If } u = Kf = \int K(x,y)f(y)d\mu(y), \text{ then } \|u\|^2 = \int Kf(x)\overline{f(x)}d\mu(x).$$

Thus  $K(x,y)$  is a pseudo-reproducing kernel if and only if it is the kernel of an integral operator which is positive in the most general sense, i. e. on the class of linear combinations of charac-

1. The proof is not obvious. It will be given in the development of the general theory of measurable spaces and pseudo-reproducing kernels.

teristic functions of measurable sets of finite measure.<sup>1</sup> The corresponding  $\mu$ -measurable functional Hilbert space is the completion of the range of the integral operator in the norm (3.2). Theorem 1 has the following corollary about positive integral operators.

Corollary 1. If  $K$  is a positive integral operator defined on the class of linear combinations of characteristic functions of sets of finite measure, then the kernel of  $K$  is non-negative almost everywhere if and only if it is real and for each real  $u$  in the range of  $K$  there exists  $\tilde{u}$  in the completion of the range such that  $\tilde{u}(x) \geq |u(x)|$  almost everywhere and  $\|\tilde{u}\| \leq \|u\|$ . The norm in question is that defined in (3.2).

#### 4. Green's functions.

The integral operators in which we are chiefly interested are the Green's functions for differential systems. Let  $A$  be a linear elliptic self-adjoint differential operator defined in a bounded domain  $D$ , and let  $\{\hat{B}_i\}$  be a normal system of boundary operators such that the system  $(A; \{\hat{B}_i\})$  is self-adjoint.<sup>2</sup> The differential system  $(A; \{\hat{B}_i\})$  defines a symmetric operator, which also will be called  $A$ , on the Hilbert space  $L^2$  of functions which are square integrable over  $D$ . The domain of the operator  $A$  is the set of all sufficiently regular functions  $u$  on  $\bar{D}$  which satisfy

1. We treat the complex case. In the real case it must be assumed that  $K(x, y) = K(y, x)$ .

2. We use the terminology introduced in N. Aronszajn and A. N. Milgram [4].

the boundary conditions  $\hat{B}_i u = 0$ . If the closure of the operator  $A$  has a bounded inverse defined everywhere on  $L^2$ , and if the inverse is an integral operator, then the kernel  $G(x,y)$  of this integral operator is called the Green's function of the system  $(A; \{\hat{B}_i\})$ . It is clear that if  $u = Gf$ , then  $\int_D Gf(x)\overline{f(x)} dx = \int_D u(x)\overline{Au(x)} dx$ , so that the quadratic form in (3.2) is non-negative if and only if <sup>1</sup>.

$$(4.1) \quad \|u\|^2 = \int_D Au(x)\overline{u(x)} dx ,$$

is non-negative for all  $u$  in the domain of  $A$ . Since it is assumed that the closure of  $A$  has a bounded inverse, if (4.1) is non-negative, there exists a constant  $c > 0$  such that

$$\|u\|^2 \geq c \int_D |u|^2 dx \text{ for } u \text{ in the domain of } A;$$

that is, the differential system  $(A; \{\hat{B}_i\})$  is positive definite.

In classical usage the term Green's function is restricted to kernels  $G(x,y)$  which are sufficiently regular: if the order of the operator  $A$  is  $2m$ , if its coefficients are sufficiently differentiable, and if the boundary  $\partial D$  is sufficiently smooth, then, for fixed  $y \in \overline{D}$ ,  $G(x,y)$ , as a function of  $x$ , should be at least of class  $C^{2m}$  in  $\overline{D} - \{y\}$ . Recent advances in the theory of elliptic partial differential equations have made it possible to prove, in a wide variety of cases, not only the existence of the Green's function as we have defined it, but also the regularity required in the classical definition. The proof, especially the proof of regularity, is

1. In order to reduce the number of notations we write  $A$  instead of the closure of  $A$  in some of the formulas.

based on the property of coerciveness, which has to do with the relation between the quadratic form in (4.1) and the standard m-norm, defined as follows

$$(4.2) \quad \|u\|_m^2 = \sum_{k=0}^m \sum_{k_1+\dots+k_n \leq m-k} \int_D \binom{k}{k_1, \dots, k_n} \left| \frac{\partial^k u}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right|^2 dx.$$

The quadratic form in (4.1) is said to be coercive on the domain of A if there exist constants  $c \geq 0$  and  $N > 0$  such that

$$(4.3) \quad \|u\| + c \|u\|_{m-1} \geq N \|u\|_m \quad \text{for all } u \text{ in the domain of } A. \quad 1.$$

If the quadratic form (4.1) is coercive on the domain of A, the following statement can be proved.

If the coefficients of A are of class  $C^m$ , then the existence of the Green's function  $G(x,y)$  in the sense given above is equivalent to the fact that 0 is not an eigenvalue of the closure of A.  
Furthermore, if  $G(x,y)$  exists, the integral operator with kernel  $G(x,y)$  is completely continuous in the space  $L^2$ . <sup>2.</sup>

1. The first coerciveness inequality was proved by Gårding [9] for the case of Dirichlet boundary conditions. The general notion of coerciveness was introduced and investigated by N. Aronszajn [3]. A more complete presentation of the subject will be given by the same author in a forthcoming paper.

2. If we assume the coefficients of A to be of sufficiently high Class  $C^N$ , it can be proved that  $G(x,y)$  exists in the classical sense. A proof of this fact will be published elsewhere. For special types of boundary conditions, including the Dirichlet boundary conditions, a proof of the regularity was given by Nirenberg [10]. A sketchy proof of the statement in the text was given in [11].

If the Green's function of the system  $(A; \{\hat{B}_i\})$  exists and if the system is positive definite, then we see from the last section that the Green's function is a pseudo-reproducing kernel. The corresponding functional Hilbert space is the completion of the domain of  $A$  in the norm (4.1). If the norm (4.1) is coercive, it can be proved that the domain of  $A$  has a perfect functional completion, the functions of which a.e. derivatives in the ordinary sense of orders  $\leq m$  in  $D$  and of orders  $\leq m-1$  on the boundary of  $D$ . This completion contains all functions of class  $C^{(m-1,1)}$  (i.e. functions of class  $C^{m-1}$  with Lipschitz  $(m-1)^{st}$  derivatives) which satisfy the stable boundary conditions, that is those of order  $\leq m-1$ . Therefore, Theorem 1 or Corollary 1 give a necessary and sufficient condition in order that the Green's function be non-negative.

It is clear that the foregoing considerations are valid for differential systems considered in a relatively compact subdomain of a differentiable manifold.

##### 5. Green's functions for differential problems of order 2.

We consider a real elliptic linear differential operator  $A$  of order 2 on an oriented differentiable manifold  $M^n$  of class  $C^3$ . Such an operator (or its negative) is expressible in each coordinate patch in the form

$$Au = - \sum a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \underline{\text{terms of lower order}}$$

where  $a^{ij}$  is a real symmetric contravariant tensor of rank 2

which forms a positive definite matrix at every point. It is assumed that the tensor  $a^{ij}$  is of class  $C^2$ , that the coefficients of the first derivatives of  $u$  (which do not form a tensor) are of class  $C^1$ , and that the coefficient of  $u$  is continuous. It is assumed further that the operator  $A$  is self-adjoint with respect to some positive density  $\rho(x)dx^1 \dots dx^n$  of class  $C^2$ .<sup>1</sup>

If  $\{a_{ij}\}$  denotes the matrix inverse to  $\{a^{ij}\}$  and if  $a$  denotes the determinant of  $\{a_{ij}\}$ , then  $g_{ij} = (\frac{\rho}{\sqrt{a}})^2 a_{ij}$  is a Riemannian metric on  $M^n$ . Henceforth  $M^n$  is considered as a Riemannian manifold with this metric.<sup>2</sup> If  $g$  denotes the determinant of  $g_{ij}$  then  $\sqrt{g} = \rho$ , so  $A$  is expressible in any coordinate patch in the form

$$Au = - \sum \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \sqrt{g} a^{ij} \frac{\partial u}{\partial x^i} + au$$

where  $a$  is a continuous real valued function.

Let  $D$  be a relatively compact domain in  $M^n$ , and let  $M^{n-1}$  be a portion of the boundary,  $\partial D$ , which is a submanifold of  $M^n$  of class  $C^2$ . It can be proved that the most general normal system of boundary operators at  $M^{n-1}$  which is self-adjoint relative to  $A$  is a system composed of a single operator, either the operator

1. This means that  $\rho(x)dx^1 \dots dx^n$  is an exterior differential form of rank  $n$ , that in each coordinate patch  $\rho(x)$  is positive and of class  $C^2$ , and that for every pair of functions  $u$  and  $v$  of class  $C^2$  and 0 outside a compact set,  $\int Au\bar{v} \rho(x) dx^1 \dots dx^n = \int u\bar{A}v \rho(x) dx^1 \dots dx^n$ .

2. We have assumed that the coefficients of  $A$  are somewhat more regular than is strictly necessary, in order to be able to use this metric, thereby simplifying a number of the formulas considerably.



$\hat{B}_0 u = u$  or the operator  $\hat{B}_1 u = \frac{\partial u}{\partial \nu} + bu$ , where  $b$  is an arbitrary continuous real valued function on  $M^{n-1}$ , and where  $\frac{\partial}{\partial \nu}$  is the interior normal derivative in the metric  $g_{ij}$ .<sup>1</sup>

It is assumed that the boundary  $\partial D$  is a submanifold of  $M^n$  of class  $C^{(0,1)}$ , and that  $\partial D$  is also the boundary of  $M^n - \bar{D}$ . It is assumed in addition that  $\partial D$  is piecewise of class  $C^2$  in the following sense: there exist a finite number of disjoint submanifolds  $M_i^{n-1}$  of  $M^n$  such that  $\partial D = \bigcup \overline{M_i^{n-1}}$ ; for each  $i$ ,  $\overline{M_i^{n-1}}$  is contained in a submanifold  $\tilde{M}_i^{n-1}$  of  $M^n$  of class  $C^2$ ; and in the usual measure on  $\tilde{M}_i^{n-1}$ ,  $\overline{M_i^{n-1}} - M_i^{n-1}$  has measure 0.

Let  $\hat{B}$  be a boundary operator at  $\partial D$  obtained by choosing for each  $M_i^{n-1}$  either the operator  $\hat{B}_0$  or one of the operators  $\hat{B}_1$ , (with  $b$  continuous on  $\overline{M_i^{n-1}}$ ) and consider the bilinear form

$$(5.1) \quad Q(u, v) = \int_D a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \sqrt{g} \, dx + \int_D au \bar{v} \sqrt{g} \, dx - \int_{\partial D} \rho(s) ku \bar{v} \, ds$$

where  $k$  is the function on  $\partial D$  which on  $M_i^{n-1}$  is equal to 0 if the boundary operator chosen on  $M_i^{n-1}$  is  $\hat{B}_0$  and is equal to  $-b$  if the boundary operator chosen on  $M_i^{n-1}$  is  $\hat{B}_1$ , and where  $\rho(s)$  is the value of the characteristic polynomial of  $A$  for the unit normal to  $\partial D$  at the point  $s$ . If  $u$  and  $v$  are functions of class  $C^2$  in a

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1. The terminology used here is that introduced in N. Aronszajn and A. N. Milgram [4]; and all of the calculations used in finding the form of self-adjoint systems are based on the general theory of differential operators on Riemannian manifolds developed there. The operator  $\hat{B}_1$  had a particularly simple form because of the special choice of the Riemannian metric on  $M^n$ .

neighborhood of  $\bar{D}$  and if  $\hat{B}u = \hat{B}v = 0$ , then

$$(5.2) \quad Q(u,v) = \int_D Au \bar{v} \sqrt{g} \, dx.$$

The quadratic form  $Q(u,u)$  is easily proved to be coercive, even on the class of all functions, a fortiori on the domain of  $A$ . Therefore, if the differential system  $(A;\hat{B})$  is positive definite, then the Green's function exists and is a pseudo-reproducing kernel for the completion of the domain of  $A$  with the norm  $\sqrt{Q(u,u)}$ . In view of the statements in the last section about this functional space, the scalar product in the complete space is still given by (5.1). Furthermore, the class of Lipschitz functions satisfying the stable boundary conditions (i.e.  $\hat{B}_0 u = 0$  whenever  $\hat{B} = \hat{B}_0$ ) forms a dense subspace of the completion. For a function  $u$  of this class,  $|u|$  is clearly also of this class, and can be taken as the function  $\tilde{u}$  in condition (b) of Theorem 1, since  $Q(|u|, |u|) = Q(u,u)$ . This leads to

Theorem 2. The three statements below are equivalent.

- (i) The system  $(A;\hat{B})$  is positive definite.
- (ii) The Green's function for the system  $(A;\hat{B})$  exists and is a pseudo-reproducing kernel.
- (iii) The Green's function for the system  $(A;\hat{B})$  exists and is non-negative.

Proof. The equivalence of statements (i) and (ii) and the fact that they imply statement (iii) are already proved. We assume therefore, that the system  $(A;B)$  is not positive definite, but that the Green's function  $G(x,y)$  exists, and we show that  $G(x,y)$  cannot

be non-negative.

According to the part of the theorem which is proved, the Green's function  $G_\rho(x,y)$  for the system  $(A+\rho; \hat{B})$  exists and is non-negative, provided the number  $\rho > 0$  is sufficiently large.  $G_\rho$  defines a completely continuous integral operator on  $L^2$ . Let  $\frac{1}{\mu}$  be its largest eigenvalue. By a well-known theorem of Jentsch,  $\frac{1}{\mu}$  is positive and simple and the corresponding eigenfunction  $f$  is non-negative. It is easily verified that  $\mu - \rho \leq 0$  and that  $f = (\mu - \rho)Gf$ . Obviously, therefore,  $G$  cannot be non-negative.

#### 6. Green's functions for differential problems of order 4.

The examples of Duffin and Garabedian mentioned in the introduction show that there is no condition resembling the conditions (i) and (ii) of Theorem 2 which always will ensure that the Green's function of a problem of order 4 is positive. In this section we do not attempt to treat the general question; we only present some interesting special cases.

We assume that  $M^n$  is an oriented differentiable manifold of sufficiently high class, that  $D$  is a domain in  $M^n$  with a sufficiently regular boundary, and that  $A$  is a second order differential operator of the type considered in the last section with sufficiently regular coefficients.  $M^n$  is given a Riemannian metric as before. We consider differential systems of the form  $(A^2; \hat{B}_0, \hat{B}_2)$  and  $(A^2; \hat{B}_1, \hat{B}_3)$ , where

$$\hat{B}_0 u = u ,$$

$$\hat{B}_1 u = \frac{\partial u}{\partial \nu} + bu ,$$

$$\hat{B}_2 u = \frac{-c}{\rho(s)^2} \frac{\partial u}{\partial \nu} + \frac{1}{\rho(s)} Au ,$$

$$\hat{B}_3 u = \frac{d}{\rho(s)^2} u + \frac{1}{\rho(s)} \hat{B}_1 Au ,$$

in which  $b$ ,  $c$ , and  $d$  are sufficiently regular functions on  $\partial D$ . Letting  $A_0$  be the operator on  $L^2$  corresponding to the first system, and  $A_1$  be the operator corresponding to the second, we define

$$Q_0(u, u) = \int_D |Au|^2 \sqrt{g} \, dx - \int_{\partial D} c \left| \frac{\partial u}{\partial \nu} \right|^2 ds ,$$

$$Q_1(u, u) = \int_D |Au|^2 \sqrt{g} \, dx - \int_{\partial D} d |u|^2 ds .$$

If  $u$  belongs to the domain of  $A_0$ , then  $\int_D A^2 u \bar{u} \, dx = Q_0(u, u)$  and if  $u$  belongs to the domain of  $A_1$ , then  $\int_D A^2 u \bar{u} \, dx = Q_1(u, u)$ . Furthermore, it is not difficult to prove that  $Q_0(u, u)$  is coercive on the domain of  $A_0$  and  $Q_1(u, u)$  is coercive on the domain of  $A_1$ . We get the following

Theorem 3. If both systems  $(A; B_0)$  and  $(A^2; \hat{B}_0, \hat{B}_2)$  are positive definite and if  $c \geq 0$ , then the Green's function for  $(A^2; \hat{B}_0, \hat{B}_2)$  exists and is non-negative. If both systems  $(A; \hat{B}_1)$  and  $(A^2; \hat{B}_1, \hat{B}_3)$  are positive definite and if  $d \geq 0$ , then the Green's function for  $(A^2; \hat{B}_1, \hat{B}_3)$  exists and is non-negative.

Proof. Only the first statement will be proved; the second is proved similarly.

The domain of  $A_0$  with the norm  $\sqrt{Q_0(u, u)}$  has a functional completion whose elements are the functions of the form

$u(x) = G_0 f(x)$ , where  $G_0$  is the Green's function for the system  $(A; \hat{B}_0)$  and where  $f$  is square integrable. Clearly

$$Q_0(u, u) = \int_D |f|^2 \sqrt{g} \, dx - \int_{\partial D} c \left| \frac{\partial u}{\partial \nu} \right|^2 \, ds.$$

Since  $G_0 \geq 0$ ,  $\frac{\partial G_0}{\partial \nu} \geq 0$ . Hence the function  $\tilde{u}(x) = G_0 |f|(x)$  has the properties  $\tilde{u}(x) \geq |u(x)|$  and  $\frac{\partial \tilde{u}}{\partial \nu}(x) \geq \left| \frac{\partial u}{\partial \nu}(x) \right|$ . Since also  $c \geq 0$  it follows that  $Q_0(\tilde{u}, \tilde{u}) \leq Q_0(u, u)$ .

Remark. If  $(A; \hat{B}_0)$  is positive definite, then, for sufficiently small  $c$ ,  $(A^2; \hat{B}_0, \hat{B}_2)$  is also positive definite. If  $c = 0$ , the statement of the last theorem is trivial, since the Green's function for the system  $(A^2; \hat{B}_0, \frac{1}{\rho(s)} A)$  is  $\int_D G_0(x, z) G_0(z, y) \sqrt{g} \, dz$ . A similar remark holds for the system  $(A^2; \hat{B}_1, \hat{B}_3)$  and the function d.

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